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Contractions on Hilbert space

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Let T be a contraction ,that is $\|T\| \leq 1$, on a separable Hilbert space $\mathcal{H}$. Then $D_T = (I-T^*T)^{1/2}$ is well defined, which is called defect operator of T. In this case we have $\sigma(T) \subset \tilde{D}$, where D and $\tilde{D}$ denote the open unit disc and its closure respectively. Contractions which have defect operators of finite ranks have been studied by many mathematicians. For investigations of contraction T with $D_T \in (\sigma,c)$, that is $I-T^*T \in (\tau,c)$, where (σ,c) and (τ,c) denote the Hilbert Schmidt class and the trace class respectively, some mathematicians added a condition $\sigma(T) \neq \tilde{D}$. Such a contraction T was called weak contraction by M.G.Krein. The spectral decomposition for weak contraction T or accretive operator

$$(I+T)(I-T)^{-1}$$

were obtained by Sz.-Nagy and Foias,Brodskii and Ginzburg (cf .[7]).

Since T is a contraction, $\|T^n x\|$ is decreasing for each x. Sz.-Nagy and Foias defined contractions' classes as following:

$$C_{1\cdot} = \{T: \lim_{n\to\infty} \|T^n x\| > 0 \text{ for each } x \neq 0\},$$

$$C_{0\cdot} = \{T: \lim_{n\to\infty} \|T^n x\| = 0 \text{ for each } x\},$$

$$C_{\cdot 1} = \{T: T^* \in C_{1\cdot}\}, \quad C_{\cdot 0} = \{T: T^* \in C_{0\cdot}\},$$

$$C_{ij} = C_{i\cdot} \cap C_{\cdot j} \quad (0 \leq i,j \leq 1).$$

These formal notations are playing important roles in the studies of contraction. In particular they showed that every weak contraction in C_{00} belongs to C_0 (about this notation see [7]), and every weak contraction is decomposed to direct sum of the contraction in C_0 and the contraction in C_{11} . The Jordan models for weak contractions were constructed by P.Y.Wu [10].

In [9] the author applied the results of Bercovici and Voiculescu's paper [1] to investigate a contraction T satisfying $\sigma(T) = \tilde{D}$ and $D_T \in (\sigma, c)$, in particular, showed that T belongs to C_{10} iff there is a quasi-affinity X such that

$$X\ T = S_E\ X\ ,$$

where E is a Hilbert space with dim E = - index T (this "index" is Fredfolm index) and S_E is the unilateral shift on $\ell^2_+(E)$. From the results of [9], he conjectured that contraction in C_{00} with (σ,c)-defect operator belongs to C_0 . In [8] Takahashi and Uchiyama showed that this was true.

In this note we will clear the structure of a contraction T with D_T in (σ,c). In particular, setting

$\alpha = \min\{\dim N(T-\lambda) : \lambda \in D\}$, $\beta = \min\{\dim N(T^*-\lambda) : \lambda \in D\}$,

where $N(T) = \{x : Tx = 0\}$, we will show that there are vector valued holomorphic functions $h_i(\lambda)$, $f_j(\lambda)$ $(1 \le i \le \alpha, 1 \le j \le \beta)$ defined on D satisfying

$$(T-\lambda)h_i(\lambda) \equiv 0 \quad , \quad (T^*-\lambda)\ f_j(\lambda) \equiv 0$$

, and that if $\alpha = \beta = 0$, then T is a weak contraction.

In section 4, we will study the weighted shifts with finite matrices' weights.

From now on, we use the symbol D(T) instead of D_T for convenience

1. Upper triangulation

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then, since $\sum_i (1-\|Te_i\|^2)<\infty$ for a C.O.N.B. $\{e_i\}$ of $\mathcal{H}$, we have $\dim N(T)<\infty$. Let $T=V|T|$ be the polar decomposition of T. Then there is a isometric (or co-isometric) extension V_1 of V such that V_1-V is of finite rank. In this case $\dim N(V_1-\lambda)$ is constant on D and finite, also $\dim N(V_1{}^*-\lambda)$ is constant on D. Since range$(V_1-\lambda)$ is closed $(V_1-\lambda)$ is a semi-Fredholm operator, and index$(V_1-\lambda)$ is constant on D.
Since $T-\lambda=V_1-\lambda+(V-V_1)-V(I-|T|)$, $T-\lambda$ is a semi-Fredholm operator, and index$(T-\lambda)$ is constant on D and less than ∞. Thus we have

$$(1.1)\qquad \sigma(T)\cap D = \{\sigma_p(T)\cup\overline{\sigma_p(T^*)}\}\cap D \quad .$$

Now we notice that if $\dim N(T^*)$ is finite, then $(T-\lambda)$ is a Fredholm operator for each $\lambda\in D$.

From the definition of C_1. it follows that

$$(1.2)\qquad \sigma_p(T)\cap D = \phi \qquad \text{for } T\in C_1. \quad .$$

In this section we obtain an upper triangulation of T whose diagonal elements were already studied.

The next lemma is trivial, but for the sake of the completeness we prove it.

Lemma 1.1. Let Y be a bounded operator and F a Fredholm operator such that $FY \in (\tau,c)$. Then we have $Y \in (\tau,c)$.

Proof. There are bounded operators F' and P such that

$$F'F = I-P \quad , \text{range } P = N(F).$$

Thus $(I-P)Y = F'FY \in (\tau,c)$ implies $Y = (I-P)Y+PY \in (\tau,c)$. Q.E.D.

Lemma 1.2. Let T be a contraction with $D(T) \in (\sigma,c)$ and let

$$T = \begin{bmatrix} T_{0\cdot} & B \\ 0 & T_{1\cdot} \end{bmatrix} \tag{1.3}$$

be the decomposition of T such that $T_{0\cdot} \in C_{0\cdot}$, $T_{1\cdot} \in C_{1\cdot}$ (see[7]). Then $D(T_{0\cdot})$ and $D(T_{1\cdot})$ are in (σ,c) and B in (τ,c).

Proof. Since $I-T^*T \in (\tau,c)$,

$I-T_{0\cdot}^* T_{0\cdot}$, $B^*T_{0\cdot}$ and $I-(B^*B+T_{1\cdot}^*T_{1\cdot})$

belong to (τ,c), where I of "$I-T_{0\cdot}^* T_{0\cdot}$" is the identity on the space where $T_{0\cdot}$ is defined. From next lemma , it follows that $T_{0\cdot}$ is a Fredholm operator. Thus , by Lemma 1.1, we have $B \in (\tau,c)$ and hence $I-T_{1\cdot}^*T_{1\cdot} \in (\tau,c)$. Q.E.D.

Lemma 1.3. Suppose $T_{0\cdot} \in C_{0\cdot}$ and $D(T_{0\cdot}) \in (\sigma,c)$, then $T_{0\cdot}$ is a Fredholm operator.

Proof. Let

$$T_{0\cdot} = \begin{bmatrix} T_{01} & A \\ 0 & T_0 \end{bmatrix} \tag{1.4}$$

be the decomposition of $T_{0\cdot}$ satisfying $T_{01}\in C_{01}$ and $T_0\in C_{00}$([7]). Since $I-T_{0\cdot}{}^*T_{0\cdot}\in(\tau,c)$, $I-T_{01}{}^*T_{01}$, A^*T_{01} and $I-(A^*A+T_0{}^*T_0)$ are in (τ,c) too. From (1.2) we have $\sigma_p(T_{01}{}^*)\cap D=\phi$,hence T_{01} is a Fredholm operator. Consequently, from Lemma 1.1 ,$A\in(\tau,c)$ and hence $I-T_0{}^*T_0\in(\tau,c)$.Since $T_0\in C_{00}$, we have $T_0\in C_0$ [8], which implies $\dim N(T_0) = \dim N(T_0{}^*)<\infty$ [7].Therefore T_0 is a Fredholm operator. Thus

$$T_{0\cdot} = \begin{bmatrix} T_{01} & 0 \\ 0 & T_0 \end{bmatrix} + \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

is a Fredholm operator. Q.E.D.

Lemma 1.4. Suppose $T_{1\cdot}\in C_{1\cdot}$ and $D(T_{1\cdot})\in(\sigma,c)$ and let

$$T_{1\cdot} = \begin{bmatrix} T_{11} & F \\ 0 & T_{\cdot 0} \end{bmatrix}$$

be a decomposition of $T_{1\cdot}$ such that $T_{11}\in C_{11}$, $T_{\cdot 0}\in C_{\cdot 0}$([7]). Then $D(T_{11})$ and $D(T_{\cdot 0})$ are in (σ,c) and F in (τ,c),and $T_{\cdot 0}\in C_{10}$.

Proof. $I-T_{11}{}^*T_{11}$, F^*T_{11} and $I-(F^*F+T_{\cdot 0}{}^*T_{\cdot 0})$belong to (τ,c). From (1.2) we have

$$\sigma_p(T_{11})\cap D = \phi \quad\text{and}\quad \sigma_p(T_{11}{}^*)\cap D = \phi ,$$

and hence , by (1.1) we have

$$\sigma(T_{11})\cap D = \phi . \tag{1.5}$$

Thus $F\in(\tau,c)$ and hence $I-T_{\cdot 0}{}^*T_{\cdot 0}\in(\tau,c)$. To show $T_{\cdot 0}\in C_{10}$, decompose $T_{\cdot 0}$ as

(1.6) $$T_{\cdot 0} = \begin{bmatrix} T_{00} & F_3 \\ 0 & T_{10} \end{bmatrix},$$

where $T_{00} \in C_{00}$, $T_{10} \in C_{10}$. Then we have $I - T_{00}{}^* T_{00} \in (\tau, c)$ and hence $T_{00} \in C_0$, from which we get

(1.7) $$\sigma(T_{00}) \cap D \neq D \quad .$$

Denote the space on which $T_{1\cdot}$ is defined by $\mathcal{L}$, and let $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ be a decomposition of $\mathcal{L}$ corresponding to

$$T_{1\cdot} = \begin{bmatrix} T_{11} & F_1 & F_2 \\ 0 & T_{00} & F_3 \\ 0 & 0 & T_{10} \end{bmatrix},$$

where $[\ F_1\ ,\ F_2\ ] = F$. Set

(1.8) $$T_2 = \begin{bmatrix} T_{11} & F_1 \\ 0 & T_{00} \end{bmatrix} .$$

Then, since $T_2 = T_{1\cdot} | \mathcal{L}_1 \oplus \mathcal{L}_2$, we have $T_2 \in C_{1\cdot}$ and $D(T_2) \in (\sigma, c)$. Above triangulation of T_2 implies that

$$\sigma(T_2) \subset \sigma(T_{11}) \cup \sigma(T_{00}) \ .$$

From this relation and (1.5), (1.7), it follows that

$$\sigma(T_2) \cap D \neq D \quad .$$

Therefore T_2 is a weak contraction. The $C_0 - C_{11}$ decomposition of T_2 ([7]) implies T_2 has no C_0-part , because $T_2 \in C_{1\cdot}$, and so $T_2 \in C_{11}$. From (1.8) we have $T_{00}{}^* = T_2{}^* | \mathcal{L}_2$,which belongs to $C_{0\cdot}$ and $C_{1\cdot}$; this is impossible. Thus $\mathcal{L}_2$ reduces to 0, so that from (1.6) we have $T_{\cdot 0} = T_{10} \in C_{10}$. Q.E.D.

Theorem 1.5. Let T be a contraction with $D(T) \in (\sigma,c)$. Then we have an upper triangulation :

$$T = \begin{bmatrix} T_{01} & & * & \\ 0 & T_0 & & \\ 0 & 0 & T_{11} & \\ 0 & 0 & 0 & T_{10} \end{bmatrix},$$

where $D(T_{01})$, $D(T_0)$, $D(T_{11})$ and $D(T_{10})$ belong to (σ,c), and $T_{01} \in C_{01}$, $T_0 \in C_0$, $T_{11} \in C_{11}$, $T_{10} \in C_{10}$, and * belongs to (τ,c).

Proof. At first, decompose T as Lemma 1.2 ,next decompose $T_{0\cdot}$ as (1.4). In the proof of Lemma 1.3 we showed that T_{01} and T_0 satisfy the conditions in theorem. At last decompose $T_{1\cdot}$ as Lemma 1.4 and set $T_{10} = T_{\cdot 0}$. Q.E.D.

Definition. Above upper triangulation is called the canonical triangulation for T with $D(T) \in (\sigma,c)$.

Remark. We showed that T_{01} and T_0 are Fredholm operators and T_{11} is invertible. But dim $N(T_{10}{}^*)$ may be infinite.

2. Eigenvectors

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Set

$$\alpha = \min\{\dim N(T-\lambda) : \lambda \in D\}, \quad \beta = \min\{\dim N(T^*-\lambda) : \lambda \in D\},$$

$$i(\lambda) = \dim N(T-\lambda) - \alpha \quad (<\infty), \quad \Lambda = \{\lambda \in D : i(\lambda) > 0\}.$$

Now we note that if a bounded operator A is decomposed as

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \text{ where } A_1 \text{ is a surjection,}$$

then $\dim N(A) = \dim N(A_1) + \dim N(A_3)$. In fact, we have

$$N(A) = N(A_1) + \{(-B^{-1}A_2x, x) : x \in N(A_3)\},$$

where B is the restriction of A_1 to $N(A_1)^{\perp}$.

Theorem 2.1. Let T be a contraction with $D(T) \in (\sigma, c)$. And consider the canonical triangulation of T. Then

$$\alpha = \dim N(T_{01}) \quad \text{and} \quad \beta = \dim N(T_{10}^*).$$

Proof. At first, we notice (1.3). Since $\sigma_p(T_{1\cdot}) \cap D = \phi$, it is not difficult to show $N(T-\lambda) = N(T_{0\cdot}-\lambda)$ for $\lambda \in D$. Next we notice (1.4). Since $D(T_{01}) \in (\sigma, c)$ and $\sigma_p(T_{01}^*) \cap D = \phi$, $(T_{01}-\lambda)$ is a surjection for each $\lambda \in D$. Thus we have

$$(2.1) \qquad \dim N(T-\lambda) = \dim N(T_{0\cdot}-\lambda) = \dim N(T_{01}-\lambda) + \dim N(T_0-\lambda)$$

$$= \text{index } (T_{01}-\lambda) + \dim N(T_0-\lambda) = \text{index } T_{01} + \dim N(T_0-\lambda).$$

$T_0 \in C_0$ implies that $\sigma(T_0) \cap D$ is countable. Hence we have

$$\alpha = \text{index } T_{01} = \dim N(T_{01}).$$

To show $\beta = \dim N(T_{10}^*)$, take the adjoint of (1.3), that is

$$T^* = \begin{bmatrix} T_1.^* & B^* \\ 0 & T_0.^* \end{bmatrix} .$$

Since $\sigma_p(T_1.)\cap D = \phi$ and $D(T_1.)\in(\sigma, c)$, $(T_1.^*-\lambda)$ is a surjection for each $\lambda\in D$. Thus we have

$$\dim N(T^*-\lambda) = \dim N(T_1.^*-\lambda) + \dim N(T_0.^*-\lambda) .$$

From (1.4), it follows that $N(T_0.^*-\lambda) = N(T_0{}^*-\lambda)$ for $\lambda\in D$, because $\sigma_p(T_{01}{}^*)\cap D = \phi$. Now we notice the decomposition of $T_1.$ in Lemma 1.4 and remark that we set T_{10} instead of $T._0$ in the canonical triangulation of T. Since $\sigma_p(T_{11}{}^*)\cap D = \phi$, it is clear that $N(T_1.^*-\lambda) = N(T_{10}{}^*-\lambda)$ for $\lambda\in D$, so that

$$\dim N(T^*-\lambda) = \dim N(T_{10}{}^*-\lambda) + \dim N(T_0{}^*-\lambda).$$

Consequently we have $\beta = \dim N(T_{10}{}^*)$. Q.E.D.

Corollary 2.2. Let T be a contraction with $D(T)\in(\sigma, c)$. Then $\sum_{\lambda\in\Lambda} (1-|\lambda|)\cdot i(\lambda) < \infty$.

Proof. From (2.1), we have $i(\lambda) = \dim N(T_0-\lambda)$. Thus, by [7] we can conclude the proof. Q.E.D.

Theorem 2.3. Let T be a contraction with $D(T)\in(\sigma, c)$. Then there are holomorphic vector valued functions $h_i(\lambda)$, $f_j(\lambda)$, $(1\le i\le\alpha$, $1\le j\le\beta)$ defined on D such that

$$(T - \lambda)\ h_i(\lambda) \equiv 0 \quad (T^*-\lambda)\ f_j(\lambda) \equiv 0 ,$$

and for each $\lambda\in D$ $\{h_1(\lambda),\dots,h_\alpha(\lambda)\}$ are linearly independent, also $\{f_1(\lambda),\dots,f_\beta(\lambda)\}$ are. In this case, setting

$\mathcal{L}^{\perp} = \bigvee \{h_i(\lambda), f_j(\lambda) : i, j, \lambda\}$, $P_{\mathcal{L}} T|_{\mathcal{L}}$ is a weak contraction.

Proof. We showed that $T_{0\perp}$ in the canonical triangulation of T is a Fredholm operator. Hence

$$T_{01}^*(I - T_{01}T_{01}^*) = (I - T_{01}^*T_{01})T_{01}^* \in (\tau, c)$$

implies, by Lemma 1.1, $D(T_{01}^*) \in (\sigma, c)$. Therefore there is a quasi-affinity X such that $X\, T_{01}^* = S_E X$, where $\dim E = -\text{index}\, T_{01}^* = \dim N(T_{01}) = \alpha < \infty$ [9]. Let $\{e_1, \ldots, e_\alpha\}$ be a C.O.N.B. of E . Then $g_i(\lambda) = \{e_i, \lambda e_i, \lambda^2 e_i, \ldots\}$ $(1 \le i \le \alpha)$ is holomorphic function defined on D with value in $\ell_+^2(E)$. And for each $\lambda \in D$ $\{g_1(\lambda), \ldots, g_\alpha(\lambda)\}$ are orthogonal each other. It is trivial to show that

$$(S_E^* - \lambda) g_i(\lambda) \equiv 0 , \quad \bigvee_{i\,\lambda} g_i(\lambda) = \ell_+^2(E) .$$

Since $T_{01}X^* = X^*S_E^*$,

$$h_i(\lambda) = \begin{bmatrix} X^* g_i(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1 \le i \le \alpha)$$

satisfy the conditions given in the theorem. Since $T_{10} \in C_{10}$ and $D(T_{10}) \in (\sigma, c)$, there is a quasi-affinity Y such that

$$Y\, T_{10} = S_F\, Y \quad \text{, where } \dim F = \beta \le \infty .$$

We can show the existence of $f_j(\lambda)$ with the same way as above ; hence we omit it. We must show the last assertion. To this end, we notice that $\{h_i(\lambda) : 1 \le i \le \alpha , \lambda \in D\}$ and $\{f_j(\lambda) : 1 \le j \le \beta , \lambda \in D\}$ span the spaces on which T_{01} and T_{10}, respectively, are defined . Thus , by Theorem 1.5 we have

$$P_{\mathcal{L}}\, T|_{\mathcal{L}} = \begin{bmatrix} T_0 & * \\ 0 & T_{11} \end{bmatrix}. \tag{2.2}$$

In this case * clearly belongs to (τ,c). Now we set $T_{\mathcal{L}}=P_{\mathcal{L}}T|_{\mathcal{L}}$. From (2.2) ,$D(T_0)\in(\sigma,c)$ and $D(T_{11})\in(\sigma,c)$ imply that $D(T_{\mathcal{L}})\in(\sigma,c)$. Since T_{11} is invertible , we have

$$\sigma_p(T_{\mathcal{L}})=\sigma_p(T_0) \qquad \sigma_p(T_{\mathcal{L}}^*)=\sigma_p(T_0^*) .$$

$T_0\in C_0$ implies that $\sigma_p(T_0^*)=\overline{\sigma_p(T_0)}\neq D$ [7]. Thus by(1.1) we have $\sigma(T_{\mathcal{L}})\cap D=\sigma_p(T_0) = \Lambda\neq D$. Thus $T_{\mathcal{L}}$ is a weak contraction. Q.E.D.

Theorem 2.4. Let T be a contraction with $D(T)\in(\sigma,c)$;then the following are equivalents:

(a) $\alpha=\beta=0$;

(b) T is a weak contraction ;

(c) T is decomposable (about definition see [2]).

Proof. (a)$\Rightarrow$(b): From Theorem 2.1. $N(T_{01})= 0$,which implies T_{01} is a weak contraction. Therefore there is a C_0-C_{11} decomposition of T_{01} , but it is impossible , because $T_{01}\in C_{01}$. Thus the space on which T_{01} is defined reduces to 0. Similarly the space on which T_{10} is defined reduces to 0. Thus $\mathcal{L}$ in Theorem 2.3 is $\mathcal{H}$. Therefore T is a weak contraction.

(b)$\Rightarrow$(c): This was shown by Jafarian [5].

(c)$\Rightarrow$(a): Since decomposable T has the single valued extension property, $\alpha=0$ follows . Thus for $\lambda \notin \Lambda$, $(T-\lambda)$ is injective semi-Fredholm operator. Hence $\sigma_{\ell}(T) \cap D \subset \Lambda$. Thus we have $\sigma(T) \cap D \subset \Lambda$ (see p.30 of [2]). Consequently $\beta = 0$. Q.E.D.

Proposition 2.5. Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then $T \in C_{10}$ if and only if there are vector valued holomorphic functions $h_i(\lambda)$ such that

$$(T^* - \lambda) h_i(\lambda) \equiv 0 \ , \quad \bigvee_{i,\lambda} h_i(\lambda) = \mathcal{H} \ .$$

Proof. "Only if" part follows from Theorem 2.3 and its proof. We must show "if" part. Since

$$T^{*n} h_i(\lambda) = \lambda^n h_i(\lambda) \to 0 \quad \text{as } n \to \infty,$$

T^{*n} strongly converges to 0 on linear spann of $\{h_i(\lambda) : i, \lambda\}$. Suppose

$$T^{*n} x_i \to 0 \quad (n \to \infty) \text{ and } x_i \to x \ (i \to \infty).$$

Since $\|T^{*n} x\| \le \|T^{*n} x_i\| + \|T^{*n}(x - x_i)\| \le \|T^{*n} x_i\| + \|x - x_i\|$, we have $\overline{\lim_{n\to\infty}} \|T^{*n} x\| \le \|x - x_i\|$. Since we can make the right side arbitrary small, $T^{*n} x \to 0$ $(n \to \infty)$. Thus T belongs to $C_{.0}$, therefore the canonical triangulation of T becomes

$$T = \begin{bmatrix} T_0 & * \\ 0 & T_{10} \end{bmatrix} .$$

Let P be the orthogonal projection to the space which T_0 is defined on . Then we have

$$0 = P\,(T^* - \lambda)\, h_i(\lambda) = P\,(T^* - \lambda) P\, h_i(\lambda) = (T_0^* - \lambda) P\, h_i(\lambda) .$$

Since $\sigma_p(T_0{}^*)$ are countable , $Ph_i(\lambda)\equiv 0$. Consequently $P\mathcal{H}= 0$ and hence $T=T_{10}$. Q.E.D.

Alternately we have

Proposition 2.6. Let T be a contraction on $\mathcal{H}$ with $D(T)\in(\sigma,c)$. Then $T\in C_{01}$ iff there are vector valued holomorphic functions $f_j(\lambda)$ defined on D such that

$$(T-\lambda)\ f_j(\lambda)\equiv 0 \qquad \bigvee_{j,\lambda} f_j(\lambda) = \mathcal{H} \qquad .$$

3. m-accretive operators

Let A be an m-accretive operator densely defined in $\mathcal{H}$ (about the definition see [6]). Then

$$(3.1) \qquad T = (A-I)(A+I)^{-1}$$

is a contraction defined on $\mathcal{H}$ and

$$\sigma_p(T) \not\ni 1 \qquad \text{and} \qquad T^* = (A^*-I)(A^*+I)^{-1}$$

(see Chap IV of [7]). It is trivial to show that

$$((I-T^*T)h,h) = 4\ \mathrm{Re}\ (A(A+I)^{-1}h,\ (A+I)^{-1}h) \quad \text{for } h\in\mathcal{H}.$$

Since $A(A+I)^{-1}$ and $(A+I)^{-1}$ are bounded , we have a relation:

$$I-T^*T \in (\tau,c) \Longleftrightarrow u(A) \in (\tau,c) \ ,$$

where $u(A)=\mathrm{Re}((A^*+I)^{-1}A(A+I)^{-1})$. In this section we denote the open right half plane by Ω . The mapping

$$\psi : \mu \longrightarrow \frac{\mu - 1}{\mu + 1}$$

transforms Ω onto D. It is clear that

(3.2) $\quad (A-\mu)x=0 \Longleftrightarrow (T-\psi(\mu))(A+I)x=0$.

Set

$$\alpha = \min\{\dim N(A-\mu) : \mu \in \Omega\}, \ \beta = \min\{\dim N(A^*-\mu) : \mu \in \Omega\} ,$$

$$i(\mu) = \dim N(A-\mu) - \alpha , \quad \Gamma = \{\mu : i(\mu) > 0\} .$$

Proposition 3.1. Let A be an m-accretive operator densely defined in $\mathcal{H}$. If $u(A) \in (\tau, c)$, then it follows that

$$\sum_{\mu \in \Gamma} \left(\frac{\operatorname{Re} \mu}{1+|\mu|^2}\right) \cdot i(\mu) < \infty .$$

Proof. Since range of $(A+I)$ is $\mathcal{H}$, by (3.2), we have

$$\dim N(A-\mu) = \dim N(T-\psi(\mu)), \ \alpha = \min\{\dim N(T-\lambda) : \lambda \in D\},$$

$$\dim N(T-\lambda) - \alpha = \dim N(A-\psi^{-1}(\lambda)) - \alpha = i(\psi^{-1}(\lambda)) ,$$

$$\{\lambda : i(\psi^{-1}(\lambda)) > 0\} = \psi(\Gamma).$$

Thus from Corollary 2.2 , it follows that

$$\sum_{\lambda \in \psi(\Gamma)} (1-|\lambda|) \cdot i(\psi^{-1}(\lambda)) < \infty$$

so that $\displaystyle\sum_{\mu \in \Gamma} (1-|\psi(\mu)|) \cdot i(\mu) < \infty$.

Therefore we have

$$\sum_{\mu \in \Gamma} \frac{\operatorname{Re} \mu}{1+|\mu|^2} \cdot i(\mu) < \infty \quad \text{(cf. p.132 of [4])}.$$

Theorem 3.2. Let A be an m-accretive operator densely defined in $\mathcal{H}$. If $u(A) \in (\tau, c)$, then there are vector valued holomorphic functions $x_i(\mu)$, $y_j(\mu)$, $(1\leq i\leq\alpha, 1\leq j\leq\beta)$ defined on Ω such that

$$(A-\mu)\, x_i(\mu) \equiv 0 \quad \text{and} \quad (A^*-\mu)\, y_j(\mu) \equiv 0 \quad .$$

Proof. From Theorem 2.3, for T defined by (3.1) there are holomorphic functions $h_i(\lambda)$ $(1\leq i\leq\alpha)$ such that

$$(T-\lambda)\, h_i(\lambda) \equiv 0 \ .$$

Then $$x_i(\mu) = (A+I)^{-1} h_i(\psi(\mu))$$

is a holomorphic function defined on Ω ,and for each $\mu\in\Omega$ $x_i(\mu)$ belongs the domain of A. From (3.2), we have

$$(A-\mu)\, x_i(\mu) \equiv 0.$$

We can similarly show the existence of $y_j(\mu)$ from the alternate relation of (3.2), that is

$$(A^*-\mu)x = 0 \iff (T^*-\psi(\mu))(A^*+I)x = 0 \ . \qquad \text{Q.E.D.}$$

4. Weighted unilateral shifts

In this section we study weighted unilateral shifts with (σ,c)- defect operators. Let E be an N-dimensional finite Hilbert space, and A_n (n=0.1.2...) invertible contraction on E. Let T be a weighted unilateral shift on $\ell^2_+(E)$ defined by

$$T\,\{x_0, x_1, \ldots\} = \{0, A_0x_0, A_1x_1, \ldots\}$$

Lemma 4.1. Let B be an invertible operator on E. Then we have

$$\|B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|} \quad , \quad \frac{1}{|\det B|} \leq \|B^{-1}\|^{N} \quad .$$

Proof. Let $\lambda_1 \geq \ldots\ldots \geq \lambda_N > 0$ be eigen values of B^*B. Then we have

$$\|B^{-1}\|^2 = \|(B^*B)^{-1}\| = \frac{1}{\lambda_N} \leq \frac{\lambda_1^{N-1}}{\lambda_1 \ldots \lambda_N} = \frac{\|B^*B\|^{N-1}}{\det (B^*B)} \quad .$$

Thus we have
$$\|B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|} \quad .$$

The second inequality similarly follows (cf. p.200 of [3]).Q.E.D.

Now we remember next fact:

for scalar a_n such that $0 < |a_n| < 1$, $\prod_{n=0}^{\infty} |a_n|$ converges iff $\sum_{n=0}^{\infty} (1-|a_n|) < \infty$.

Theorem 4.2. Let T be a contractive weighted shift defined above. Then the following are equivalents :

(a) $T \in C_{10}$;

(b) $D(T) \in (\sigma, c)$;

(c) T is similar with simple sift S_E ;

(d) there is a $\delta > 0$ such that

$$\|A_n \cdots A_0 x\| \geq \delta \|x\| \text{ for every } x \in E \text{ and every } n \ .$$

Proof. (d)$\Rightarrow$(c): For each m we have

$$\|A_{m+n} \cdots A_m x\| = \|A_{m+n} \cdots A_m A_{m-1} \cdots A_0 (A_{m-1} \cdots A_0)^{-1} x\|$$
$$\geq \delta \|(A_{m-1} \cdots A_0)^{-1} x\| \geq \delta \frac{1}{\|A_m \cdots A_0\|} \|x\| \geq \delta \|x\| \ ,$$

because each A_i is a contraction. Thus for each $f \in \ell_+^2(E)$, we have

$$\| T^n f \| \geq \delta \|f\| \qquad \text{for every } n.$$

By the well known Sz.-Nagy's theorem, T is similar with an isometry V. Since T belongs to $C_{\cdot 0}$, so do V, hence V is a unilateral shift. Since

$$\dim N(V^*) = \dim N(T^*) = \dim E = N$$

dimension of the wandering space for V is N. Thus V is unitarily equivalent with S_E.

(c)$\Longrightarrow$(a): This is obvious.

(a)$\Longrightarrow$(d): Set $\quad \ell(x) = \lim_{n\to\infty} \| T^n \{x,0,0,..\} \| \qquad$ for $x \in E$.

Since ℓ is continuous and $\ell(x) \neq 0$ for $x \neq 0$, there is a $\delta > 0$ such that

$$\ell(x) \geq \delta \qquad \text{for } x \text{ in the unit surface of } E.$$

Since $\ell(\alpha x) = |\alpha| \ell(x)$, we have

$$\lim_{n\to\infty} \| A_n \cdots A_0 \, x \| = \ell(x) \geq \delta \| x \| \quad \text{for } x \in E.$$

(b)$\Longrightarrow$(d): From

$$\infty > \|I - T^*T\|_1 = \sum_{n=0}^{\infty} \| I - A_n{}^*A_n \|_1 \geq \sum_{n=0}^{\infty} \| I - A_n{}^*A_n \| ,$$

it follows that

$$\prod_{n=0}^{\infty} (1 - \| I - A_n{}^*A_n \|)$$

converges and we denote its limit by δ^2. In view of

$$\| A_i^{-1} \|^2 = \|(A_i{}^*A_i)^{-1}\| = \| (I - (I - A_i{}^*A_i))^{-1} \| \leq \frac{1}{1 - \| I - A_i{}^*A_i \|} ,$$

we have

$$\| A_n \cdots A_0 \; x \; \|^2 \geq \frac{\|x\|^2}{\|(A_n \cdots A_0)^{-1}\|^2} \geq \frac{\|x\|^2}{\|A_n^{-1}\|^2 \cdots \|A_0^{-1}\|^2}$$

$$\geq \prod_{i=0}^{n} (1-\|I-A_i{}^*A_i\|) \; \|x\|^2 \geq \delta^2 \|x\|^2 \text{ for every } n.$$

(d)$\Longrightarrow$(b): Since each A_n is an invertible contractive matrix,

we have $\| 1-A_n{}^*A_n\| = 1-\min\{\lambda: \lambda\in\sigma_p(A_n{}^*A_n)\}$

$$= 1- \frac{1}{\|(A_n{}^*A_n)^{-1}\|} = 1-\frac{1}{\|A_n^{-1}\|^2} \leq 2(1-\frac{1}{\|A_n^{-1}\|})$$

from Lemma 4.1,

$$\leq 2(1-\frac{|\det A_n|}{\|A_n\|^{N-1}}) \leq 2(1-|\det A_n|).$$

From (d) and Lemma 4.1, we have

$$|\det A_n| \; \ldots\ldots \; |\det A_0| = |\det (A_n \ldots A_0)|$$

$$\geq \|(A_n \ldots A_0)^{-1}\|^{-N} \geq \delta^N,$$

which implies that $\prod_{n=0}^{\infty} |\det A_n|$ converges, and hence

$$\sum_{n=0}^{\infty} \|I- A_n{}^*A_n\| \leq 2 \sum_{n=0}^{\infty} (1-|\det A_n|) < \infty \qquad . \text{ Q.E.D.}$$

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